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Abstract Valuations: A Novel Representation of Plotkin Power Domain and Vietoris Hyperspace

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Abstract

Abstract valuations on a topological space X are functions that map open sets to 0, 1, or one value in between. We define a space of abstract valuations which for a continuous dcpo X is homeomorphic to the Plotkin power domain of X, and for a Hausdorff space X yields the Vietoris hyperspace of X. Thus we obtain a novel concrete representation of the Plotkin power domain. This representation is more similar to the standard representation of the probabilistic power domain than the previously known ones.

1 Introduction

The Plotkin power domain construction was defined by Plotkin in [5]. This first presentation was later streamlined by Smyth [1] and Plotkin himself [6]. These papers define the Plotkin power domain for ω-algebraic domains, using three different, yet equivalent concrete representations. In the first representation, the power domain $\mathcal{P}D$ of an ω-algebraic domain D is the ideal completion of the set of non-empty finite subsets of the basis of D, preordered by the Egli-Milner ordering: $F \sqsubseteq_{EM} G$ iff $F \subseteq {\downarrow}G$ and $G \subseteq {\uparrow}F$. In the second representation, $\mathcal{P}D$ consists of equivalence classes of *finitely generable* subsets of D preordered by a topological variant of the Egli-Milner ordering: $A \sqsubseteq_{TEM} B$ iff $A \subseteq \mathsf{cl}\, B$ and $B \subseteq {\uparrow}A$. In the third representation, $\mathcal{P}D$ is the set of *lenses* of D, again ordered by '$\sqsubseteq_{TEM}$'. Here, a lens is a non-empty subset of D which arises as the intersection of a Scott closed and a compact upper set.

Abramsky and Jung [4] summarise the knowledge about the Plotkin power domain in the more general case of continuous domains. The basis representation easily generalises to this case; if D is a continuous domain with basis

B, then $\mathcal{P}D$ is a continuous domain which is the rounded ideal completion of the set of non-empty finite subsets of B equipped with the way-below version of the Egli-Milner ordering: $F \ll_{EM} G$ iff $F \subseteq \Downarrow G$ and $G \subseteq \Uparrow F$. However, the representation using lenses works for ω-continuous domains and for coherent domains, but does not work for all continuous domains; there even is an algebraic counterexample.

Quite a different description of the Plotkin power domain is given by Hennessy and Plotkin [10]: $\mathcal{P}D$ is the free semilattice over D in the category of dcpo's and continuous functions. In this manner, the power domain construction can be extended to all dcpo's. However, this is not a concrete representation: while it provides useful information about the algebraic properties of $\mathcal{P}D$, it does not give details about its internal structure.

In [8], the author derived two further concrete representations of the Plotkin power domain $\mathcal{P}D$. The first consists of certain pairs (C, K) of a closed subset C and a compact upper subset K of D. The second consists of certain functions $P : [D \to \mathbf{B}] \to \mathbf{B}$ where $\mathbf{B}$ is the Boolean domain $\{\perp, \mathit{false}, \mathit{true}\}$. The conditions which characterise the pairs / functions in $\mathcal{P}D$ are complex. For continuous dcpo's, these representations coincide with those introduced above. There are non-continuous dcpo's where they do not produce the free semilattice.

Summarising, one can say that no simple concrete representation of the Plotkin power domain was found so far. The situation is quite different with the *probabilistic power domain* of Jones and Plotkin [9]. It was already defined with a satisfactory concrete representation: the probabilistic power domain of D is the set of all probabilistic valuations on D. These are continuous, strict, and modular functions from ΩD, the lattice of open sets of D, to the unit interval $[0..1]$ of the real line.

In this paper, we present a new concrete representation of the Plotkin power domain which is simpler and more similar to the probabilistic power domain than the existing ones. It uses *abstract valuations* which differ from the valuations introduced above in that all real numbers between 0 and 1 are collapsed into a single object.

In Section 2, we define abstract valuations on an arbitrary topological space X. In Section 3, we introduce the topological space $\mathcal{P}_V X$ of abstract valuations on X. In Section 4, we make the construction $\mathcal{P}_V$ into a monad on the category of topological spaces and continuous functions. In Section 5, we show that for Hausdorff spaces X, $\mathcal{P}_V X$ is homeomorphic to the Vietoris space of non-empty compact subsets of X. In Section 6, we prove that for a continuous dcpo X (with its Scott topology), $\mathcal{P}_V X$ is isomorphic to the Plotkin power domain of X. The proof uses the description of the Plotkin power domain via its basis.

2 Abstract Valuations

An abstract valuation differs from a valuation by the fact that all numbers in between 0 and 1 are replaced by one abstract value. Thus, let $\mathbf{A}$ be the dcpo of three elements 0, M, and 1, ordered by $0 \sqsubset \mathsf{M} \sqsubset 1$. Often, we shall have to compare two elements of $\mathbf{A}$.

Proposition 2.1 *For a, b in $\mathbf{A}$, $a \sqsubseteq b$ iff $a = 1 \Rightarrow b = 1$ and $b = 0 \Rightarrow a = 0$.*

Now, we could proceed by defining abstract valuations formally, and study their properties. To exhibit the intuitions behind their definition, we defer the formal definition until 2.2, and first derive some $\mathbf{A}$-valued functions from subsets of X. The properties of these functions will motivate the definition of abstract valuations.

2.1 Functions Derived from Subsets

For a non-empty subset A of X, we define a function A^* from the opens of X to $\mathbf{A}$ as follows: $A^*(U)$ is 0 if $A \cap U = \emptyset$, it is 1 if $A \cap U = A$ (i.e., $A \subseteq U$), and it is M otherwise. Non-emptiness of A is needed to guarantee that $A \cap U = \emptyset$ and $A \subseteq U$ cannot occur together. On the other hand, $\emptyset^*$ cannot be defined consistently.

Proposition 2.2 *For a non-empty subset A of X, A^* has the following properties:*

(i) *Monotonicity: $U \subseteq V \Rightarrow A^*(U) \sqsubseteq A^*(V)$;*

(ii) *Strictness $A^*(\emptyset) = 0$ and normalisation $A^*(X) = 1$;*

(iii) *0-property: If $A^*(U) = 0$, then $A^*(U \cup V) = A^*(V)$ for all opens V.*
1-property: If $A^*(U) = 1$, then $A^*(U \cap V) = A^*(V)$ for all opens V.
We use the name 0-1-property *when referring to both properties together.*

(iv) *If $A^*(U_i) = 0$ holds for all i in I, then $A^*(\bigcup_{i \in I} U_i) = 0$ follows.*

Proof. We concentrate on (iii). Property $A^*(U) = 0$ means $A \cap U = \emptyset$ and thus $A \cap (U \cup V) = A \cap V$. Property $A^*(U) = 1$ means $A \cap U = A$ and thus $A \cap (U \cap V) = A \cap V$. □

We are particularly interested in Scott continuous functions A^*. By properties (i) and (iv) of Prop. 2.2, A^* is Scott continuous once for every directed family $(U_i)_{i \in I}$ of opens, $A^*(\bigcup_{i \in I} U_i) = 1$ implies $A^*(U_i) = 1$ for some i in I. By definition of A^*, this means that $A \subseteq \bigcup_{i \in I} U_i$ implies $A \subseteq U_i$ for some i. This is exactly compactness of A.

Proposition 2.3 *Let A be a non-empty subset of X. Then A^* is Scott continuous iff A is compact.*

2.2 The Definition of Abstract Valuations

As the defining properties of abstract valuations, we take the properties of Prop. 2.2 plus continuity. We shall soon see that part (iv) of Prop. 2.2 is redundant; thus, it is not included in the definition.

Definition 2.4 *An* abstract valuation *(or shortly* $\boldsymbol{A}$*-valuation) on a topological space* X *is a function* α *from the dcpo* ΩX *of opens of* X *to* $\boldsymbol{A}$ *with the following properties:*

(i) *Continuity:* $\alpha(\bigcup_{i\in I} U_i) = \bigsqcup_{i\in I} \alpha U_i$ *for every directed family* $(U_i)_{i\in I}$ *of opens of* X*.*

(ii) *Strictness* $\alpha\emptyset = 0$ *and normalisation* $\alpha X = 1$*.*

(iii) *Insignificance: If* $\alpha U = 0$*, then* $\alpha(U \cup V) = \alpha V$ *for all opens* V*; and if* $\alpha U = 1$*, then* $\alpha(U \cap V) = \alpha V$ *for all opens* V*.*

As usual, continuity implies monotonicity, i.e., $U \subseteq V$ implies $\alpha U \sqsubseteq \alpha V$ for all open sets U and V.

Proposition 2.5 *Let* α *be an abstract valuation. If* $\alpha U = \alpha V = 1$*, then* $\alpha(U \cap V) = 1$*. For arbitrary families* $(U_i)_{i\in I}$ *of opens,* $\alpha U_i = 0$ *for all* i *in* I *implies* $\alpha(\bigcup_{i\in I} U_i) = 0$*.*

Proof. The property for intersection and an analogous property for union are special instances of 0-1-property. With strictness and Scott continuity, the statement about arbitrary unions follows. □

2.3 Abstract Valuations Induced by Subsets

By the results collected so far, every non-empty compact subset K of a topological space X induces an abstract valuation K^* on X. In general, the relationship between non-empty compact subsets and **A**-valuations is complex because of the following two reasons:

- Different subsets may induce the same abstract valuation; this holds in particular for all subsets with the same convex hull (see also Prop. 3.4 below).
- Even in algebraic domains, there are abstract valuations which cannot be obtained from a non-empty compact set by the star operator.

2.4 Sums of Abstract Valuations

Now, we define a binary operation '+' on abstract valuations. We first define '+' for **A**, and then lift it to **A**-valuations. The motivation for our definition of the sum comes from considering $(K \cup L)^*$ for non-empty compact subsets K and L on X. Obviously, $(K \cup L) \cap U = \emptyset$ holds iff $K \cap U = L \cap U = \emptyset$, and $K \cup L \subseteq U$ iff $K \subseteq U$ and $L \subseteq U$. Thus, we define:

For n and m in $\mathbf{A}$, $n+m$ is 0 if $n=m=0$; it is 1 if $n=m=1$, and M otherwise.

Note that in particular, $0+1=\mathsf{M}$ holds; 0 is not a neutral element of '+'. With a straightforward case analysis, one can prove that '+' is monotonic and thus continuous since $\mathbf{A}$ is finite.

An alternative way to define '+' is to say that it is the least upper bound operation of the poset $\{0 < \mathsf{M} > 1\}$. Hence, '+' is a semilattice operation, i.e., commutative, associative, and idempotent.

We extend '+' to $\mathbf{A}$-valuations by defining $(\alpha+\beta)(U) = \alpha U + \beta U$. It is easy to prove that $\alpha+\beta$ is again an abstract valuation. By our derivation of the definition of '+', $(K \cup L)^* = K^* + L^*$ holds for all non-empty compact subsets K and L of X.

2.5 Comparing $\boldsymbol{A}$-valuations with $\boldsymbol{R}_+$-valuations

Since $\mathbf{A}$-valuations are normalised by definition, we compare them with normalised real-valued valuations (shortly $\mathbf{R}_+$-valuations). A normalised $\mathbf{R}_+$-valuation on a space X is a function ν from ΩX to $\mathbf{R}_+$ with the following properties:

(i) Scott continuity;

(ii) Strictness $\nu\emptyset = 0$ and normalisation $\nu X = 1$;

(iii) Modularity $\nu(U \cup V) + \nu(U \cap V) = \nu U + \nu V$.

Properties (i) and (ii) are in analogy to the first two properties of $\mathbf{A}$-valuations, but (iii) is different. Thus, we may ask what the relationship between modularity and 0-1-property is. The answer depends on the value domain: for $\mathbf{R}_+$-valued functions, modularity is strictly stronger than 0-1-property, while for $\mathbf{A}$-valuations, 0-1-property is strictly stronger than modularity. In particular, it follows that normalised $\mathbf{R}_+$-valuations enjoy the 0-1-property, and $\mathbf{A}$-valuations are modular.

Proposition 2.6 *Let $\nu : \Omega X \to \boldsymbol{R}_+$ be monotonic and normalised. If ν is modular, then it enjoys the 0-1-property. The opposite implication does not hold.*

Proof. Assume ν is modular. If $\nu U = 0$, then $\nu(U \cap V)$ is also 0 by monotonicity. Hence, $\nu(U \cup V) = \nu(U \cup V) + \nu(U \cap V) = \nu U + \nu V = \nu V$. If $\nu U = 1$, then $\nu(U \cup V)$ is also 1 by monotonicity and normalisation. Hence, $\nu(U \cap V) = \nu(U \cup V) + \nu(U \cap V) - 1 = \nu U + \nu V - 1 = \nu V$.

Let X be the discrete space $\{0, 1\}$. Let $\nu\emptyset = 0$, $\nu\{0\} = \nu\{1\} = 1/3$, and $\nu X = 1$. This function is continuous, strict, normalised, and satisfies the 0-1-property. It is not modular. □

Proposition 2.7 *Let $\alpha : \Omega X \to \boldsymbol{A}$ be monotonic. If α satisfies the 0-1-property, then it is modular. The opposite implication does not hold.*

Proof. Assume α satisfies the 0-1-property. If $\alpha U = 0$, then $\alpha(U \cap V)$ is also 0 by monotonicity, and $\alpha(U \cup V) = \alpha V$ holds. Hence, $\alpha(U \cup V) + \alpha(U \cap V) = \alpha V + 0 = \alpha U + \alpha V$. The case $\alpha U = 1$ is similar. In the case $\alpha U = \mathsf{M}$, $\alpha U + \alpha V = \mathsf{M}$ holds. By monotonicity, $\alpha(U \cap V)$ cannot be 1, and $\alpha(U \cup V)$ cannot be 0. Thus, their sum is also M.

Let X be the discrete space $\{0, 1\}$. Let $\alpha\emptyset = \alpha\{0\} = 0$, $\alpha\{1\} = \mathsf{M}$, and $\alpha X = 1$. This function is continuous, strict, normalised, and modular, but does not satisfy the 0-1-property. $\square$

3 The Space of Abstract Valuations

3.1 The Topology

Now, we define the space $\mathcal{P}_{\mathrm{V}}X$ of $\mathbf{A}$-valuations on a topological space X. Since the elements of $\mathcal{P}_{\mathrm{V}}X$ are certain functions from ΩX to $\mathbf{A}$, we may topologize $\mathcal{P}_{\mathrm{V}}X$ as a subspace of the function space $[\Omega X \to \mathbf{A}]_{\mathrm{p}}$, equipped with the pointwise topology. By this definition, the sum $+ : \mathcal{P}_{\mathrm{V}}X \times \mathcal{P}_{\mathrm{V}}X \to \mathcal{P}_{\mathrm{V}}X$ becomes a continuous semilattice operation.

There are other choices for the topology of $\mathcal{P}_{\mathrm{V}}X$, e.g., the compact-open topology or the Isbell topology, but the pointwise topology is simpler and gives the results we look for.

By the definition of the pointwise topology, the subbasic open sets of $\mathcal{P}_{\mathrm{V}}X$ are the sets $\langle U \to V \rangle = \{\alpha \in \mathcal{P}_{\mathrm{V}}X \mid \alpha U \in V\}$, where U ranges over the points of ΩX, i.e., the opens of X, and V over the opens of $\mathbf{A} = \{0 \sqsubset \mathsf{M} \sqsubset 1\}$. Because of its simple structure, $\mathbf{A}$ has merely two non-trivial open sets: $\{1\}$ and $\{\mathsf{M}, 1\}$. Thus, we can partition the non-trivial subbasic open sets of $\mathcal{P}_{\mathrm{V}}X$ into two classes: the sets $\Box U = \{\alpha \in \mathcal{P}_{\mathrm{V}}X \mid \alpha U = 1\}$, and the sets $\Diamond U = \{\alpha \in \mathcal{P}_{\mathrm{V}}X \mid \alpha U \neq 0\}$, where in both cases U ranges over the opens of X. For abstract valuations coming from non-empty compact subsets, these two classes of subbasic opens are familiar: $K^* \in \Box U$ iff $K \subseteq U$, and $K^* \in \Diamond U$ iff $K \cap U \neq \emptyset$. Furthermore, for every X, the two 'modalities' $\Box$ and $\Diamond$ satisfy the 'usual properties' known from the Vietoris power locale [7,2]:

Proposition 3.1 *$\Box$ and $\Diamond$ are Scott continuous.*

$\Box\emptyset = \Diamond\emptyset = \emptyset$	$\Box X = \Diamond X = \mathcal{P}_{\mathrm{V}}X$
$\Diamond(U \cup V) = \Diamond U \cup \Diamond V$	$\Box(U \cap V) = \Box U \cap \Box V$
$\Diamond U \cup \Box(U \cup V) = \Diamond U \cup \Box V$	$\Box U \cap \Diamond(U \cap V) = \Box U \cap \Diamond V$

These properties immediately follow from the defining properties of abstract valuations.

3.2 Sobriety of $\mathcal{P}_{\mathrm{V}}X$

Theorem 3.2 *For every topological space X, the space $\mathcal{P}_{\mathrm{V}}X$ is sober.*

Proof. We use two facts about sobriety (see also [3]):

(i) If Z is sober, then $[Y \to Z]_{\mathrm{p}}$ is sober (no matter what Y is).

(ii) If Y is sober, Z is $\mathcal{T}_0$, and $f, g : Y \to Z$ are continuous, then the equaliser $\{y \in Y \mid fy = gy\}$ is sober.

As a finite $\mathcal{T}_0$-space, $\mathbf{A}$ is sober, and so $[\Omega X \to \mathbf{A}]_{\mathrm{p}}$ is sober by (i). Property (i) of Def. 2.4, continuity, is already satisfied by all elements of this function space. Once the remaining properties (ii) and (iii) are formulated as continuous equations, $\mathcal{P}_{\mathrm{V}}X$ can be described as an equaliser, and its sobriety follows.

Strictness and normalisation are given by continuous equations. The implication $\alpha U = 0 \Rightarrow \alpha(U \cup V) = \alpha V$ is equivalent to the equation $\alpha U * \alpha(U \cup V) = \alpha U * \alpha V$, where $* : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ is given by $0 * b = b$ and $a * b = 1$ for $a \neq 0$; this binary function is monotonic and thus continuous. The dual implication can be handled analogously (with a different binary operation). $\Box$

3.3 The Dcpo of Abstract Valuations

Because of the pointwise topology, the specialisation preorder on $\mathcal{P}_{\mathrm{V}}X$ is pointwise: $\alpha \sqsubseteq \beta$ iff $\alpha U \sqsubseteq \beta U$ for all opens U. Since all spaces $\mathcal{P}_{\mathrm{V}}X$ are sober, we immediately obtain [11, II.3.17]:

Theorem 3.3 *For every space X, $(\mathcal{P}_{\mathrm{V}}X, \sqsubseteq)$ is a dcpo. Every continuous function $f : \mathcal{P}_{\mathrm{V}}X \to \mathcal{P}_{\mathrm{V}}Y$ is Scott continuous from $(\mathcal{P}_{\mathrm{V}}X, \sqsubseteq)$ to $(\mathcal{P}_{\mathrm{V}}Y, \sqsubseteq)$.*

From the derivation of $\mathcal{P}_{\mathrm{V}}X$ as an equaliser from a pointwise function space, it also follows that directed joins in $\mathcal{P}_{\mathrm{V}}X$ are given pointwise.

3.4 The Order in the Case of Subsets

Let A and B be two non-empty compact subsets of X. We look for a criterion for $A^* \sqsubseteq B^*$. By Prop. 2.1, $A^* \sqsubseteq B^*$ holds iff for all open sets U, $B^*(U) = 0 \Rightarrow A^*(U) = 0$ and $A^*(U) = 1 \Rightarrow B^*(U) = 1$. With the definition of the star operator and contraposition of the first implication, this becomes $A \cap U \neq \emptyset \Rightarrow B \cap U \neq \emptyset$ and $A \subseteq U \Rightarrow B \subseteq U$. By standard topological arguments, this is equivalent to $\mathsf{cl}\, A \subseteq \mathsf{cl}\, B$ and $\uparrow\! A \supseteq \uparrow\! B$, where '$\mathsf{cl}$' means topological closure, and '$\uparrow$' refers to the specialisation preorder. This can be simplified to $A \subseteq \mathsf{cl}\, B$ and $\uparrow\! A \supseteq B$, which is the topological Egli-Milner ordering.

Proposition 3.4 *For two non-empty compact subsets A and B of X, $A^* \sqsubseteq B^*$ holds in $\mathcal{P}_{\mathrm{V}}X$ iff $A \subseteq \mathsf{cl} B$ and $\uparrow\! A \supseteq B$, i.e., $A \sqsubseteq_{TEM} B$.*

For finite F, $\mathsf{cl}\, F$ is $\downarrow\! F$. Thus, we obtain the familiar Egli-Milner ordering:

Proposition 3.5 *For two non-empty finite subsets F and G of X, $F^* \sqsubseteq G^*$ holds in $\mathcal{P}_{\mathrm{V}}X$ iff $F \subseteq \downarrow\! G$ and $\uparrow\! F \supseteq G$, i.e., $A \sqsubseteq_{EM} B$.*

4 Monad Operations

In this section, we make $\mathcal{P}_{\mathrm{V}}$ into a monad in the category TOP of topological spaces and continuous functions. In turns out that proofs become simpler when a localic point of view is adopted, i.e., instead of a continuous function $f : X \to Y$, the corresponding inverse image function $f^- : \Omega Y \to \Omega X$ is considered.

4.1 The Unit of the Monad: Singletons

The unit of the monad is given by the singleton maps $\mathsf{s}_X : X \to \mathcal{P}_{\mathrm{V}} X$. They are easily defined with the star operator: $\mathsf{s}_X x = \{x\}^*$. Hence, $\mathsf{s}_X x\,(U)$ is 1 if x is in U, and is 0 otherwise.

Proposition 4.1 *For every space X and open O of X, $\mathsf{s}_X^-(\Box O) = \mathsf{s}_X^-(\Diamond O) = O$ holds. Therefore, $\mathsf{s}_X : X \to \mathcal{P}_{\mathrm{V}} X$ is continuous for all X, and moreover a topological embedding for $\mathcal{T}_0$-spaces X.*

4.2 The Functor $\mathcal{P}_{\mathrm{V}}$

For every continuous $f : X \to Y$, let $\mathcal{P}_{\mathrm{V}} f : \mathcal{P}_{\mathrm{V}} X \to \mathcal{P}_{\mathrm{V}} Y$ be defined by $\mathcal{P}_{\mathrm{V}} f(\alpha)(V) = \alpha(f^- V)$ for α in $\mathcal{P}_{\mathrm{V}} X$ and V in ΩY.

Proposition 4.2 *For continuous $f : X \to Y$ and open V of Y, the two equations $(\mathcal{P}_{\mathrm{V}} f)^-(\Box V) = \Box(f^- V)$ and $(\mathcal{P}_{\mathrm{V}} f)^-(\Diamond V) = \Diamond(f^- V)$ hold. The function $\mathcal{P}_{\mathrm{V}} f : \mathcal{P}_{\mathrm{V}} X \to \mathcal{P}_{\mathrm{V}} Y$ is well-defined, continuous, and additive. If f is a topological embedding, then so is $\mathcal{P}_{\mathrm{V}} f$. The map $\mathcal{P}_{\mathrm{V}}$ is functorial: $\mathcal{P}_{\mathrm{V}}\, \mathsf{id} = \mathsf{id}$ and $\mathcal{P}_{\mathrm{V}}(g \circ f) = \mathcal{P}_{\mathrm{V}} g \circ \mathcal{P}_{\mathrm{V}} f$ holds.*

Proof. $\mathcal{P}_{\mathrm{V}} f(\alpha) \in \Box V$ iff $\mathcal{P}_{\mathrm{V}} f(\alpha)(V) = 1$ iff $\alpha(f^- V) = 1$ iff $\alpha \in \Box(f^- V)$. The proof of the remaining claims is similarly straightforward. $\Box$

On **A**-valuations induced by non-empty compact sets, $\mathcal{P}_{\mathrm{V}} f$ operates as the direct image function f^+:

Proposition 4.3 *For every non-empty compact subset of X and continuous $f : X \to Y$, $\mathcal{P}_{\mathrm{V}} f(K^*) = (f^+ K)^*$ holds.*

Proof. $\mathcal{P}_{\mathrm{V}} f(K^*)(V) = 1$ iff $K^*(f^- V) = 1$ iff $K \subseteq f^- V$ iff $f^+ K \subseteq V$ iff $(f^+ K)^*(V) = 1$, and similarly for 0. $\Box$

Since $\mathsf{s}_X x = \{x\}^*$, we immediately obtain:

Proposition 4.4 *s is natural, i.e., for continuous $f : X \to Y$, $\mathcal{P}_{\mathrm{V}} f \circ \mathsf{s}_X = \mathsf{s}_Y \circ f$ holds.*

4.3 The Multiplication of the Monad

Now, we look for a family of maps $\mathsf{m}_X : \mathcal{P}_{\mathrm{V}}(\mathcal{P}_{\mathrm{V}} X) \to \mathcal{P}_{\mathrm{V}} X$. In the following, the index X is often omitted.

Given β in $\mathcal{P}_\mathrm{V}(\mathcal{P}_\mathrm{V}X)$ and an open U of X, we have to define $\mathsf{m}\beta(U)$ as an element of $\mathbf{A}$. From the analogy with the Vietoris power locale [7], we expect that the inverse image function m^- satisfies the equations $\mathsf{m}^-(\Box U) = \Box\Box U$ and $\mathsf{m}^-(\Diamond U) = \Diamond\Diamond U$. Hence, $\mathsf{m}\beta(U) = 1$ iff β is in $\mathsf{m}^-(\Box U) = \Box\Box U$ iff $\beta(\Box U) = 1$. Since for a in $\mathbf{A}$, $a = 1$ implies $a \neq 0$, $\Box U$ is a subset of $\Diamond U$. By monotonicity of β, $\beta(\Box U) = 1$ implies $\beta(\Diamond U) = 1$. Hence, $\beta(\Box U) = 1$ iff $\beta(\Box U) = \beta(\Diamond U) = 1$ iff $\beta(\Box U) + \beta(\Diamond U) = 1$. Similarly, $\mathsf{m}\beta(U) \neq 0$ iff $\beta(\Diamond U) \neq 0$, or with negation, $\mathsf{m}\beta(U) = 0$ iff $\beta(\Diamond U) = 0$ iff $\beta(\Box U) = \beta(\Diamond U) = 0$ iff $\beta(\Box U) + \beta(\Diamond U) = 0$. Therefore, the localic formulae can only be satisfied if m_X is defined by $\mathsf{m}_X\beta(U) \;=\; \beta(\Box U) + \beta(\Diamond U)$.

Proposition 4.5 *For every topological space X, $\mathsf{m}_X : \mathcal{P}_\mathrm{V}(\mathcal{P}_\mathrm{V}X) \to \mathcal{P}_\mathrm{V}X$ defined by $\mathsf{m}_X\beta(U) \;=\; \beta(\Box U) + \beta(\Diamond U)$ is well-defined, continuous, and additive. For all opens U of X, $\mathsf{m}^-(\Box U) = \Box\Box U$ and $\mathsf{m}^-(\Diamond U) = \Diamond\Diamond U$ holds.*

Proof. First we verify that $\mathsf{m}\beta$ really is in $\mathcal{P}_\mathrm{V}X$, using the properties of the modalities $\Box$ and $\Diamond$ (Prop. 3.1).

- $\mathsf{m}\beta$ is Scott continuous, since $\Box$, $\Diamond$, β, and '+' are Scott continuous.
- $\mathsf{m}\beta(\emptyset) = \beta(\Box\emptyset) + \beta(\Diamond\emptyset) = \beta\emptyset + \beta\emptyset = 0+0 = 0$, and similarly for $\mathsf{m}\beta(X) = 1$.
- If $\mathsf{m}\beta(U) = 0$, then $\beta(\Box U) = \beta(\Diamond U) = 0$, whence $\beta(\Diamond(U \cup V)) = \beta(\Diamond U \cup \Diamond V) = \beta(\Diamond V)$, and $\beta(\Box(U \cup V)) = \beta(\Diamond U \cup \Box(U \cup V)) = \beta(\Diamond U \cup \Box V) = \beta(\Box V)$. Combining these two equations, $\mathsf{m}\beta(U \cup V) = \mathsf{m}\beta(V)$ follows. The proof of the statement for $\mathsf{m}\beta(U) = 1$ is analogous.

Continuity and additivity of m are obvious from its definition. The derivation of the formulae for m^- from the definition of m is essentially equal to the above derivation of the definition of m from the desired properties of m^-, read backwards. $\Box$

4.4 Verification of the Monad Laws

To show that $(\mathcal{P}_\mathrm{V}, \mathsf{s}, \mathsf{m})$ forms a monad, we have to prove four functional equations, namely naturality of m and the three monad laws. The simplest way to do this is to apply the following lemma:

Lemma 4.6 *Let X be an arbitrary space and Y a $\mathcal{T}_0$-space, and let $f, g : X \to Y$ be two continuous functions. If the inverse image functions $f^-, g^- : \Omega Y \to \Omega X$ are equal, then $f = g$ follows.*

For naturality, we have to show $\mathcal{P}_\mathrm{V}f \circ \mathsf{m}_X = \mathsf{m}_Y \circ \mathcal{P}_\mathrm{V}(\mathcal{P}_\mathrm{V}f)$ for all continuous f. Using the Lemma, we compare the inverse images of the subbasic opens $\Box V$ and $\Diamond V$ where $V \in \Omega Y$. In the following inverse image calculations, we adopt the convention that application is right associative, i.e., FGX stands for $F(GX)$.

$$(\mathcal{P}_{\mathrm{V}} f \circ \mathsf{m})^{-}\Box V \;=\; \mathsf{m}^{-}(\mathcal{P}_{\mathrm{V}} f)^{-}\Box V \;=\; \mathsf{m}^{-}\Box f^{-}V \;=\; \Box\Box f^{-}V$$
$$(\mathsf{m} \circ \mathcal{P}_{\mathrm{V}}\mathcal{P}_{\mathrm{V}} f)^{-}\Box V \;=\; (\mathcal{P}_{\mathrm{V}}\mathcal{P}_{\mathrm{V}} f)^{-}\mathsf{m}^{-}\Box V \;=$$
$$(\mathcal{P}_{\mathrm{V}}\mathcal{P}_{\mathrm{V}} f)^{-}\Box\Box V \;=\; \Box(\mathcal{P}_{\mathrm{V}} f)^{-}\Box V \;=\; \Box\Box f^{-}V$$

The case of $\Diamond V$ is completely analogous.

The three monad laws are $\mathsf{m}_X \circ \mathsf{s}_{\mathcal{P}_{\mathrm{V}} X} = \mathsf{m}_X \circ \mathcal{P}_{\mathrm{V}}\mathsf{s}_X = \mathsf{id}_{\mathcal{P}_{\mathrm{V}} X} : \mathcal{P}_{\mathrm{V}} X \to \mathcal{P}_{\mathrm{V}} X$ and $\mathsf{m}_X \circ \mathsf{m}_{\mathcal{P}_{\mathrm{V}} X} = \mathsf{m}_X \circ \mathcal{P}_{\mathrm{V}}\mathsf{m}_X : (\mathcal{P}_{\mathrm{V}})^3 X \to \mathcal{P}_{\mathrm{V}} X$. Let us concentrate on the box case of the last; the remaining calculations are similarly straightforward.

$$\mathsf{m}^{-}\mathsf{m}^{-}\Box O \;=\; \mathsf{m}^{-}\Box(\Box O) \;=\; \Box\Box\Box O$$
$$(\mathcal{P}_{\mathrm{V}}\mathsf{m})^{-}\mathsf{m}^{-}\Box O \;=\; (\mathcal{P}_{\mathrm{V}}\mathsf{m})^{-}\Box\Box O \;=\; \Box\mathsf{m}^{-}\Box O \;=\; \Box\Box\Box O$$

Of course, a direct proof of these functional equations is also possible, but much more involved than the simple calculations presented here. The reader is invited to try the last one, using the definitions of $\mathcal{P}_{\mathrm{V}} f$ and m.

5 The Hausdorff Case

The goal of this section is to prove the following theorem:

Theorem 5.1 *For every Hausdorff space X, the space $\mathcal{P}_{\mathrm{V}} X$ is homeomorphic to the Vietoris hyperspace of X, i.e., the set $\mathcal{P}_{com} X$ of non-empty compact subsets of X with the topology generated from the subbasic opens $\Box O = \{K \in \mathcal{P}_{com} X \mid K \subseteq O\}$ and $\Diamond O = \{K \in \mathcal{P}_{com} X \mid K \cap O \neq \emptyset\}$, where O ranges over the opens from X.*

We already have a map from $\mathcal{P}_{com} X$ to $\mathcal{P}_{\mathrm{V}} X$, namely the star map $K \mapsto K^*$, and we know that the two topologies correspond to each other. The only thing to show is that the star map is a bijection.

For injectivity, we prove $K^* \sqsubseteq L^* \Rightarrow K \subseteq L$. By Prop. 3.4, $K^* \sqsubseteq L^*$ implies $K \subseteq \mathsf{cl}\, L$. In a Hausdorff space, all compact sets are closed, whence $K \subseteq L$ follows.

For surjectivity of $K \mapsto K^*$, let α be a member of $\mathcal{P}_{\mathrm{V}} X$. The set $\{O \in \Omega X \mid \alpha O = 1\}$ is a Scott open filter in ΩX by continuity of α and Prop. 2.5. Since Hausdorff spaces are sober, the Hoffmann-Mislove Theorem can be applied. Hence, there is a compact saturated subset K of X such that $\alpha O = 1$ iff $K \subseteq O$. If K were empty, $\alpha\emptyset = 1$ would hold contradicting strictness.

Let W be the union of all opens O with $\alpha O = 0$. By Prop. 2.5, $\alpha W = 0$ holds, and thus, $\alpha O = 0$ iff $O \subseteq W$. Let C be the complement of W. Then $\alpha O = 0$ iff $C \cap O = \emptyset$. In the following, we show $C = K$, which is sufficient to conclude $\alpha = K^*$.

For $C \subseteq K$, let x be in C and assume $x \notin K$. By the Hausdorff property, there are disjoint opens U and V such that $x \in U$ and $K \subseteq V$. Then $\alpha V = 1$, and thus $\alpha U = \alpha(U \cap V) = \alpha\emptyset = 0$, whence $x \in U \subseteq W$, contradicting $x \in C$.

For $K \subseteq C$, we show that for all opens O, $C \subseteq O$ implies $K \subseteq O$. This is sufficient since all subsets of a Hausdorff space are saturated, i.e., intersections of opens. If $C \subseteq O$, then $W \cup O = X$, and from $\alpha W = 0$, $\alpha O = \alpha(W \cup O) = \alpha X = 1$ follows, whence $K \subseteq O$.

6 The Continuous Case

Here, we show that for a continuous dcpo X with Scott topology, $\mathcal{P}_{\mathrm{V}}X$ is again continuous with Scott topology, and its basis has the same description as the basis of the Plotkin power domain.

For finite subsets F of X and abstract valuations α on X, we define a relation '$\prec$' by

$$F \prec \alpha \text{ iff } \forall a \in F : \alpha(\Uparrow a) \neq 0 \text{ and } \alpha(\Uparrow F) = 1.$$

In the sequel, we shall show several properties of this relation. In particular, we compare it with the way-below relation on $\mathcal{P}_{\mathrm{V}}X$ and the way-below version of the Egli-Milner relation on subsets of X which is defined by $A \ll_{EM} B$ iff $A \subseteq \Downarrow B$ and $\Uparrow A \supseteq B$.

(i) If $F \prec \alpha$, then F is not empty. For, $\alpha(\Uparrow F)$ is 1 while $\alpha\emptyset$ is 0.

Hence, we need not worry about non-emptiness when we construct some finite F with $F \prec \alpha$.

(ii) If $F^* \sqsubseteq G^*$, $G \prec \alpha$, and $\alpha \sqsubseteq \beta$, then $F \prec \beta$.

Proof. By Prop. 3.5, $F^* \sqsubseteq G^*$ is equivalent to $F \sqsubseteq_{EM} G$. Therefore, every a in F is below some b in G, whence $\Uparrow b \subseteq \Uparrow a$ and thus $\beta(\Uparrow a) \sqsupseteq \alpha(\Uparrow b) \neq 0$. The relation $F \sqsubseteq_{EM} G$ also implies $G \subseteq \uparrow F$, whence $\Uparrow G \subseteq \Uparrow F$. Hence, $\beta(\Uparrow F) \sqsupseteq \alpha(\Uparrow G) = 1$ holds. □

(iii) If $F \prec \alpha$, then $F^* \sqsubseteq \alpha$.

Proof. For opens U, we have to show $F^*U \neq 0 \Rightarrow \alpha U \neq 0$ and $F^*U = 1 \Rightarrow \alpha U = 1$.

If $F^*U \neq 0$, then F meets U. Let a be an element of the intersection. Since a is in U, $\Uparrow a \subseteq U$ holds, and a in F implies $\alpha(\Uparrow a) \neq 0$. With monotonicity of α, $\alpha U \neq 0$ follows.

If $F^*U = 1$, then $F \subseteq U$, whence $\Uparrow F \subseteq U$ and thus $\alpha U \sqsupseteq \alpha(\Uparrow F) = 1$. □

(iv) Every open set U with $\alpha U = 1$ has a finite subset F such that $F \prec \alpha$.

Proof. Since α is continuous, and U is the directed union of the sets $\Uparrow H$ with finite $H \subseteq U$, there is some finite subset H of U such that $\alpha(\Uparrow H) = 1$. Let $F = \{a \in H \mid \alpha(\Uparrow a) \neq 0\}$, and $G = H \setminus F$. Then $\alpha(\Uparrow b) = 0$ holds for all b in G, whence $\alpha(\Uparrow G) = 0$ by Prop. 2.5. By property (iii) of abstract valuations, $\alpha(\Uparrow F) = \alpha(\Uparrow F \cup \Uparrow G) = \alpha(\Uparrow H) = 1$ holds. Thus we obtain $F \prec \alpha$. □

(v) For every abstract valuation α on X, there is some finite $F \subseteq X$ such that $F \prec \alpha$.

Proof. From (iv) and $\alpha X = 1$. $\square$

(vi) If $F \prec \alpha$ and $G \prec \alpha$, then there is a finite subset H of X such that $F, G \ll_{EM} H \prec \alpha$.

Proof. By Prop. 2.5, $\alpha(\Uparrow F) = \alpha(\Uparrow G) = 1$ implies $\alpha(\Uparrow F \cap \Uparrow G) = 1$. By (iv), there is a finite subset H' of $\Uparrow F \cap \Uparrow G$ such that $H' \prec \alpha$.

For all a in F, $\alpha(\Uparrow a) \neq 0$ holds. Since $\alpha(\Uparrow G)$ is 1, the 1-property implies $\alpha(\Uparrow a \cap \Uparrow G) = \alpha(\Uparrow a) \neq 0$. By Prop. 2.5, there is some c_a in $\Uparrow a \cap \Uparrow G$ such that $\alpha(\Uparrow c_a) \neq 0$.

Analogously, for all b in G, we find some d_b in $\Uparrow F \cap \Uparrow b$ such that $\alpha(\Uparrow d_b) \neq 0$. Let $H = H' \cup \{c_a \mid a \in F\} \cup \{d_b \mid b \in G\}$. This is a finite subset of X. By straightforward arguments, it is shown that it has the required properties. $\square$

(vii) For α in $\mathcal{P}_{\mathrm{V}}X$, let $D = \{F^* \mid F \prec \alpha\}$. Clearly, $F \ll_{EM} G$ implies $F \sqsubseteq_{EM} G$, which is equivalent to $F^* \sqsubseteq G^*$ by Prop. 3.5. Thus (v) and (vi) imply that D is directed. We claim $\bigsqcup D = \alpha$.

Proof. The relation $\bigsqcup D \sqsubseteq \alpha$ follows from (iii). To prove the opposite relation, we have to show $\alpha U \neq 0 \Rightarrow (\bigsqcup D)(U) \neq 0$ and $\alpha U = 1 \Rightarrow (\bigsqcup D)(U) = 1$.

If $\alpha U \neq 0$, there is some a in U such that $\alpha(\Uparrow a) \neq 0$ by Prop. 2.5. By (v), there is some F with $F \prec \alpha$. Let $G = \{a\} \cup F$. Obviously, $G \prec \alpha$ holds, whence G^* is in D. Since a is in $G \cap U$, $G^* U \neq 0$ holds, whence $(\bigsqcup D)(U) \neq 0$.

If $\alpha U = 1$, then by (iv), there is some finite $F \subseteq U$ with $F \prec \alpha$. Hence, F^* is in D and thus $(\bigsqcup D)(U) \sqsupseteq F^* U = 1$. $\square$

(viii) $F \prec \alpha$ holds if and only if $F^* \ll \alpha$.

Proof. First assume $F \prec \alpha$. Let D be a directed set in $\mathcal{P}_{\mathrm{V}}X$ with $\alpha \sqsubseteq \bigsqcup D$. By (ii), $F \prec \bigsqcup D$ follows. Thus, for all a in F, $(\bigsqcup D)(\Uparrow a) \neq 0$ holds. Hence, there is some δ_a in D such that $\delta_a(\Uparrow a) \neq 0$. In addition, $(\bigsqcup D)(\Uparrow F) = 1$ holds. Hence, there is some δ' in D such that $\delta'(\Uparrow F) = 1$. Let δ be an upper bound of δ' and all the δ_a in D. Then $F \prec \delta$ holds, whence $F^* \sqsubseteq \delta$ follows by (iii).

For the opposite direction, assume $F^* \ll \alpha$. By (vii), α is the directed join of abstract valuations G^* where $G \prec \alpha$. Hence, there is some $G \prec \alpha$ with $F^* \sqsubseteq G^*$. By (ii), $F \prec \alpha$ follows. $\square$

(ix) Properties (vii) and (viii) together imply that $\mathcal{P}_{\mathrm{V}}X$ is continuous with a basis consisting of all F^* for non-empty finite sets F. We show that the topology of $\mathcal{P}_{\mathrm{V}}X$ is the Scott topology.

Proof. In a sober space, every open set is Scott open. For the opposite direction, consider a basic Scott open set $\Uparrow F^*$. By (viii), $\Uparrow F^* = \{\alpha \in \mathcal{P}_{\mathrm{V}}X \mid F \prec \alpha\} = \bigcap_{a \in F} \Diamond{\Uparrow} a \cap \Box{\Uparrow} F$ holds. □

(x) To prove the claimed coincidence with the Plotkin power domain of X, we have to show that $F^* \ll G^*$ holds if and only if $F \ll_{EM} G$.

Proof. By (viii), $F^* \ll G^*$ is equivalent with $F \prec G^*$. For a in F, $G^*(\Uparrow a) \neq 0$ holds iff $G \cap \Uparrow a \neq \emptyset$ iff there is b in G with $a \ll b$. Property $G^*(\Uparrow F) = 1$ holds iff $G \subseteq \Uparrow F$. Together, this means that $F \prec G^*$ is equivalent to $F \ll_{EM} G$. □

Summarising, we obtain:

Theorem 6.1 *For a continuous dcpo X, $\mathcal{P}_{\mathrm{V}}X$ is continuous again with a basis which consists of all abstract valuations F^* for non-empty finite subsets F of X.*

The topology of $\mathcal{P}_{\mathrm{V}}X$ is the Scott topology.

The way-below relation between an element F^* of the basis and an arbitrary abstract valuation α is given by $F^* \ll \alpha$ iff $\alpha(\Uparrow a) \neq 0$ for all a in F, and $\alpha(\Uparrow F) = 1$.

The way-below relation within the basis is given by $F^* \ll G^*$ iff $F \ll_{EM} G$.

Corollary 6.2 *For every continuous dcpo X, $\mathcal{P}_{\mathrm{V}}X$ is isomorphic to the Plotkin power domain of X.*

The semilattice operation '+' of $\mathcal{P}_{\mathrm{V}}X$ coincides with that of the Plotkin power domain on the basis since $F^* + G^* = (F \cup G)^*$. By continuity of '+', this coincidence extends to the whole power domain.

7 Future Work

It is well-known [10] that for continuous dcpo's X, a continuous additive function $F : \mathcal{P}_{\mathrm{V}}X \to \mathbf{A}$ is uniquely determined by its values on singletons. This uniqueness property should be extended to a larger class of topological spaces. A related problem is to characterise the algebras of the monad $\mathcal{P}_{\mathrm{V}}$ introduced in Section 4.

There are obvious close connections between our power space $\mathcal{P}_{\mathrm{V}}X$ and the Vietoris power locale of [7,2]. One should find out for which sober spaces / spatial locales both constructions coincide.

The analogy between $\mathcal{P}_{\mathrm{V}}X$ and the probabilistic power domain should be developed further. In particular, the analogue of what is called integration in the probabilistic case should be defined and studied.

As shown in [8], the Plotkin power domain is a subdomain of the mixed and Sandwich power domain. We conjecture that these larger power domains can be described in terms of a generalisation of abstract valuations.

References

[1] M. B. Smyth. Power domains. *Journal of Computer and System Sciences*, 16:23–36, 1978.

[2] S. Vickers. *Topology via Logic*, volume 5 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 1989.

[3] R. Heckmann. Spaces of valuations. In S. Andima, R.C. Flagg, G. Itzkowitz, Y. Kong, R. Kopperman, and P. Misra, editors, *Papers on General Topology and its Applications*, pages 174 – 200. Annals of the New York Academy of Science, volume 806, December 1996.

[4] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science, Vol. III*. Oxford University Press, 1994.

[5] G. D. Plotkin. A powerdomain construction. *SIAM Journal on Computing*, 5(3):452–487, 1976.

[6] G. D. Plotkin. Domains (post-graduate lecture notes). Technical report, Department of Computer Science, University of Edinburgh, 1983. Printed by Y. Kashiwagi and H. Kondoh in 1992.

[7] S. Vickers. Information systems for continuous posets. *Theoretical Computer Science*, 116(2):201–229, 1993.

[8] R. Heckmann. Power domains and second order predicates. *Theoretical Computer Science*, 111:59–88, 1993.

[9] C. J. Jones and G. D. Plotkin. A probabilistic powerdomain of evaluations. In *LICS '89*, pages 186–195. IEEE Computer Society Press, 1989.

[10] M. C. B. Hennessy and G. D. Plotkin. Full abstraction for a simple parallel programming language. In J. Becvar, editor, *Foundations of Computer Science*, pages 108–120. *Lecture Notes in Computer Science 74*, Springer-Verlag, 1979.

[11] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, 1980.